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1-Cohomology of Chevalley Groups

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We relate the 1-cohomology of a finite Chevalley group to induced modules and finite geometry. As a corollary, we get the 1-cohomology of most adjoint modules.

INTRODUCTION

Let Σ be an irreducible root system and G the universal Chevalley group of type Σ over the field $\mathbf{F}_q$ of q elements. In this paper we want to study the 1-cohomology $H^1(G, W)$, where W is an irreducible module for G over the algebraic closure K of $\mathbf{F}_q$. Let $V := W^*$ be the dual of W, and B a Borel subgroup of G. By [CPSK] we know that if q is "large" (depending on Σ and W) then any non-splitting extension

$$0 \to K \to E \to V \to 0$$

of KG-modules (where K is considered as trivial module) satisfies the "highest weight condition":

> E is generated (as a KG-module) by a 1-dimensional subspace fixed by B. $\quad (*)$

By Frobenius reciprocity, the condition $(*)$ is equivalent to the one that E is a quotient of the module λ_B^G induced from a 1-dimensional representation λ of B. So if for given G and V, $(*)$ holds for all extensions E of V over K, then $H^1(G, W)$ can be read off from the induced module λ_B^G. (Thereby, we could replace λ_B^G by the (possibly) smaller module λ_P^G, where P is the full stabilizer in G of the B-invariant 1-space in V; namely, $(*)$ is equivalent to the same condition with B replaced by P.

The main purpose of this paper is to verify $(*)$ in many cases not covered by the stability results of [CPSK]. In fact much of the work is devoted to the case $q \leqslant 3$. In Section 2, we study the case that V is a fundamental module, and more generally, a module whose highest weight is a multiple

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of a fundamental weight. In this case we see rather quickly that $(*)$ holds for all $q > 3$. We also settle the case $q \leqslant 3$ for the fundamental modules, by reducing to some explicit results on 1-cohomology from [JP]. (We need two more such computations, which we perform in the Appendix). In Section 3, we obtain some results in the case $q > 2$; in particular, we prove that $(*)$ always holds if q is not a prime. The idea is to reduce to the case $G = SL_2(q)$, where all projective indecomposable modules are known by [AJL].

Another consequence of $(*)$ (which provided the original motivation for this paper), is the following: For each parabolic subgroup P of G, let V_P (resp. E_P) denote the fixed space in V (resp. in E) of the unipotent radical of P. By [S-1], V_P is an irreducible module for each Levi-complement L_P of P. Thus $(*)$ implies that for each P we get an extension

$$0 \to K \to E_P \to V_P \to 0 \tag{+}$$

of L_P-modules. We are particularly interested in the case that P is a minimal parabolic. Then the "semisimple part" of L_P is isomorphic to $SL_2(q)$, and we can again apply the results of [AJL] to deduce splitting of $(+)$ in many cases. If $(+)$ splits for all minimal parabolics, then we can choose complements $\bar{V}_P$ of K in E_P for all minimal parabolics P, and also for all Borel subgroups P, such that these spaces $\bar{V}_P$ form the same incidence geometry in E as the corresponding spaces V_P in V. Thus E is generated (as a vector space) by copies of the spaces V_P (for P a minimal parabolic or Borel subgroup of G), realising their incidence geometry from V; and so E is a quotient of the "universal" module $\tilde{V}$ so generated. (This module $\tilde{V}$ was introduced in [RS-2, Section 3B] in a slightly different set-up.)

So under the above hypotheses, $H^1(G, W)$ can be read off not only from λ_B^G, but already from its much smaller quotient $\tilde{V}$. Now $\tilde{V}$ has been calculated for the "minimal" modules in [RS-1] and for the adjoint modules (not of type C_l) in [V-1]. So as a corollary we get the 1-cohomology groups of these modules. Most of them have already been computed in [CPS], [JP] and [V-2], except for the adjoint modules of type B_l, G_2, and F_4 over fields with $\leqslant 9$ elements.

This paper was partly motivated by [S-2], where the idea of studying the extension theory of irreducible KG-modules via the methods of [RS-1] is introduced.

1. Notations

The following notations will be kept fixed throughout the paper:

Σ ... an irreducible root system in the Euclidean space E^l with scalar product (,)

$\langle\ ,\ \rangle$ ... defined by $\langle\alpha, \beta\rangle = 2(\alpha, \beta)/(\beta, \beta)$ for $\alpha, \beta \in E^l$

r_α ... the reflection at α, defined by $r_\alpha(\beta) = \beta - \langle\beta, \alpha\rangle\alpha$

$\Delta = \{\alpha_1, ..., \alpha_l\}$ ... a system of simple roots, defining as usual a partial order on Σ

Σ^+ ... the set of positive roots

$\lambda_1, ..., \lambda_l$ ... the corresponding fundamental weights, defined by $\langle\lambda_i, \alpha_j\rangle = \delta_{i,j}$

X ... the weight lattice, spanned by $\lambda_1, ..., \lambda_l$

$q = p^n$ ... a power of the prime p

$k = \mathbf{F}_q$ ... the field with q elements

k^* ... the multiplicative group of k

G ... the universal Chevalley group of type Σ over k, constructed as in [St].

Then G comes with

$h_\alpha: k^* \to G$ ... coroot corresponding to $\alpha \in \Sigma$

T ... Cartan subgroup generated by the $h_\alpha(s)$, $s \in k^*$, $\alpha \in \Sigma$

U_α $(\alpha \in \Sigma)$ ... root subgroups

$B = TU$ ... Borel subgroup, with $U = O_p(B) = \prod_{\alpha>0} U_\alpha$

$x_\alpha: k \to U_\alpha$ ... additive isomorphisms satisfying Chevalley's commutator formula (see [St, p. 30, R2]) and the relations $tx_\alpha(a)t^{-1} = x_\alpha(\alpha(t)a)$ for $t \in T$, $\alpha \in k$. Thereby each $\mu \in X$ acts on T via $\mu(h_\alpha(s)) = s^{\langle\mu, \alpha\rangle}$.

For each subset J of $\Pi := \{1, ..., l\}$ we set $\Sigma_J := \Sigma \cap \langle\alpha_j : j \in J\rangle$ and $G_J := \langle U_\alpha : \alpha \in \Sigma_J\rangle$. If J is non-empty then Σ_J can naturally be considered as the root system of the Chevalley group G_J, and $\Delta_J := \{\alpha_j : j \in J\}$ is a basis of Σ_J. Set $L_J := G_J T$ and $U_J := \langle U_\alpha : \alpha \in \Sigma^+ \setminus \Sigma_J\rangle$. Then L_J (resp. U_J) is a Levi complement (resp. the unipotent radical) of the parabolic subgroup $P_J = L_J U_J$ of G.

Finally, we let $A_l(q)$, $B_l(q)$ etc. denote the *universal* Chevalley group of the corresponding type over k. The commutator subgroup of any group X is denoted X', and $C_Y(X)$ is the centralizer of X in Y.

2. Modules with Highest Weight $m\lambda_j$

In this section, $l = \operatorname{rank} G > 1$. We assume that for each irreducible root system Σ the simple roots $\alpha_1, ..., \alpha_l$ are labelled as in [Bou, Planches] except for type G_2, where we let α_1 be long. To avoid ambiguities in this labelling, we let type C_l (resp. D_l) exist only for $l > 2$ (resp. $l > 3$).

LEMMA 1. *Let* $\alpha \in \Sigma^+ \setminus \Delta$. *Then* $U_\alpha \subset U'$ *unless*

(i) $q=2$, $\Sigma \cong B_l$, *and* $\alpha = \alpha_{l-1} + \alpha_l$ *or* $\alpha_{l-1} + 2\alpha_l$

(i)′ $q=2$, $\Sigma \cong C_l$, *and* $\alpha = \alpha_{l-1} + \alpha_l$ *or* $2\alpha_{l-1} + \alpha_l$

(ii) $q=2$, $\Sigma \cong F_4$, *and* $\alpha = \alpha_2 + \alpha_3$ *or* $\alpha_2 + 2\alpha_3$

(iii) $q=2$, $\Sigma \cong G_2$, *and* α *is short, or*

(iv) $q=3$, $\Sigma \cong G_2$, *and* $\alpha = \alpha_1 + \alpha_2$ *or* $\alpha_1 + 3\alpha_2$.

Proof. We explicitly need the constants appearing in Chevalley's commutator formula (see [St, Lemma 33 and (3), p. 151]). Let $\gamma, \delta \in \Sigma$ with $|\gamma| \geqslant |\delta|$. Define the commutator $[g, h] = ghg^{-1}h^{-1}$ for $g, h \in G$, and let $c, d \in k$.

(a) If γ and δ form a simple system of type A_2, then $[x_\gamma(c), x_\delta(d)] = x_{\gamma+\delta}(cd)$; hence $[U_\gamma, U_\delta] = U_{\gamma+\delta}$.

(b) If γ and δ form a simple system of type B_2, then

$$[x_\gamma(c), x_\delta(d)] = x_{\gamma+\delta}(\pm cd)\, x_{\gamma+2\delta}(\pm cd^2).$$

One checks easily that this implies $[U_\gamma, U_\delta] = U_{\gamma+\delta} U_{\gamma+2\delta}$ if $q \neq 2$.

(c) If γ and δ form a simple system of type G_2, then

$$[x_\gamma(c), x_\delta(d)] = x_{\gamma+\delta}(\pm cd)\, x_{\gamma+2\delta}(\pm cd^2)\, x_{\gamma+3\delta}(\pm cd^3)\, x_{2\gamma+3\delta}(\pm c^2 d^3),$$

$$[x_{\gamma+3\delta}(c), x_\gamma(d)] = x_{2\gamma+3\delta}(\pm cd)$$

$$[x_{\gamma+\delta}(c), x_\delta(d)] = x_{\gamma+2\delta}(\pm 2cd)\, x_{\gamma+3\delta}(\pm 3cd^2)\, x_{2\gamma+3\delta}(\pm 3c^2 d).$$

Case 1. $\Sigma \not\cong G_2$, and $q \neq 2$ if Σ has two different root lengths.

Each $\alpha \in \Sigma^+ \setminus \Delta$ can be written as $\alpha = \alpha' + \alpha''$ with $\alpha', \alpha'' \in \Sigma^+$. If α, α', and α'' have the same length, then α' and α'' form a simple system of type A_2; so $U_\alpha = [U_{\alpha'}, U_{\alpha''}] \leqslant U'$. Thus it remains to consider the case that $q \neq 2$, and α', α'' span a subsystem of Σ of type B_2. If α is short (resp. long), then α' and α'' (esp. α' and $\alpha'' - \alpha'$) form a simple system of type B_2, hence $U_\alpha \leqslant [U_{\alpha'}, U_{\alpha''}] \leqslant U'$ (resp. $U_\alpha \leqslant [U_{\alpha'}, U_{\alpha''-\alpha'}]$) by (b). If α is long, note that we may assume that $\alpha'' - \alpha'$ is positive (interchanging α' and α'' if necessary).

Case 2. $q=2$ and $\Sigma \cong B_l$.

If $\alpha \in \Sigma^+ \setminus \Delta$ is long and $\neq \alpha_{l-1} + 2\alpha_l$, then $\alpha = \alpha' + \alpha''$ for long $\alpha', \alpha'' \in \Sigma^+$ (see [Bou, Planche II]); so $U_\alpha = [U_{\alpha'}, U_{\alpha''}] \leqslant U'$ as before. Now assume α is short and $\neq \alpha_{l-1} + \alpha_l$. Then we can find long $\gamma \in \Delta$ such that $\delta := \alpha - \gamma \in \Sigma^+$ [Bou, Planche II]. Then γ and δ form a simple system of type B_2, hence $x_\alpha(c) = [x_\gamma(\pm c), x_\delta(1)]\, x_{\gamma+2\delta}(\pm c)$ by (b). So it suffices to show $U_{\gamma+2\delta} \leqslant U'$. Now $\gamma + 2\delta \neq \alpha_{l-1} + 2\alpha_l$ (since otherwise $\alpha = \gamma + \delta =$

$(\gamma+2\delta)-\delta=\alpha_{l-1}+2\alpha_l-\delta=\alpha_{l-1}+\alpha_l$ or $=\alpha_l$, which is excluded), and so $U_{\gamma+2\delta}\leqslant U'$ by the above (since $\gamma+2\delta$ is long).

Case 2'. $q=2$ and $\Sigma\cong C_l$.

This reduces to Case 2 by the exceptional isomorphism $B_l(2)\to C_l(2)$.

Case 3. $q=2$ and $\Sigma\cong F_4$.

If $\alpha\in\Sigma^+\setminus\Delta$ is long and $\neq\alpha_2+2\alpha_3,\ \alpha_2+2\alpha_3+2\alpha_4$, then $\alpha=\alpha'+\alpha''$ for long $\alpha',\ \alpha''\in\Sigma^+$ (see [Bou, Planche VIII]); so $U_\alpha\leqslant U'$ as before. Since $\beta:=\alpha_2+2\alpha_3+2\alpha_4\in\Sigma_{\{2,3,4\}}\cong C_3$, we have also $U_\beta\leqslant U'$ by Case 2'. Finally we use the exceptional graph automorphism of $F_4(2)$ (which interchanges long and short positive roots) to conclude that also $U_\alpha\leqslant U'$ for all short $\alpha\in\Sigma^+\setminus\Delta$ different from $\alpha_2+\alpha_3$.

Case 4. $\Sigma\cong G_2$.

We let $\Delta=\{\gamma,\delta\}$ with γ long. From the second formula in (c) we get $U_{2\gamma+3\delta}\leqslant U'$. If $p>3$ (resp. $p=2$ resp. $p=3$), then the third formula in (c) gives the following congruences mod $U_{2\gamma+3\delta}$ (which is a central subgroup of U): $[U_{\gamma+\delta},U_\delta]\equiv U_{\gamma+2\delta}U_{\gamma+3\delta}$ (resp. $\equiv U_{\gamma+3\delta}$ resp. $\equiv U_{\gamma+2\delta}$). So if $p>3$ then $U_{\gamma+2\delta}$, $U_{\gamma+3\delta}$, $U_{2\gamma+3\delta}\leqslant U'$, and then also $U_{\gamma+\delta}\leqslant U'$ by the first formula in (c). This settles the case $p>3$. If $p=2$ (resp. $p=3$), then $U_{\gamma+3\delta}$, $U_{2\gamma+3\delta}\leqslant U'$ (resp. $U_{\gamma+2\delta}$, $U_{2\gamma+3\delta}\leqslant U'$), and if additionally $q=2^n>2$ (resp. $q=3^n>3$) then the first formula in (c) gives also $U_\alpha\leqslant U'$ for the remaining two $\alpha\in\Sigma^+\setminus\Delta$. ∎

For any $J\subset\Pi$, set $D_J=(G_JU_J)'$. This group will play a key role in the proof of Proposition 1 below.

LEMMA 2. *Let $J=\Pi\setminus\{i\}$ for some $i\in\Pi$. Assume rank $G>2$ if $q\leqslant 3$. Then $U_\alpha\leqslant D_J$ for all $\alpha\in\Sigma^+\setminus\Delta$, unless $q=2$, $\Sigma\cong B_l$ or C_l, and $i=l-2$.*

Proof. By Lemma 1 (and by using the isomorphism $B_l(2)\to C_l(2)$) it only remains to consider the following two cases:

Case A. $q=2$, $\Sigma\cong B_l$ $(l\geqslant 3)$, $i\neq l-2$, $\alpha=\alpha_{l-1}+\alpha_l$ or $\alpha_{l-1}+2\alpha_l$.

Since $i\neq l-2$, there is $n\in G_J$ representing the Weyl reflection $r:=r_{\alpha_{l-2}}$. Then $r(\alpha)$ is different from the two exceptional roots in Lemma 1(i), and lies in $\Sigma^+\setminus\Delta$. Hence $U_{r(\alpha)}\leqslant U'$ by Lemma 1, and so

$$U_\alpha=n^{-1}U_{r(\alpha)}n\leqslant n^{-1}U'n\leqslant D_J.$$

Case B. $q=2$, $\Sigma\cong F_4$, $\alpha=\alpha_2+\alpha_3$ or $\alpha_2+2\alpha_3$.

In this case there is $n\in G_J$ representing either r_{α_1} or r_{α_4}. Now argue as in Case A. ∎

LEMMA 3. *Let $J = \Pi \backslash \{i\}$ for some $i \in \Pi$. If $q \leqslant 3$ (resp. $q = 2$) assume that the diagram of Δ_J has no isolated node (resp. no connected component of type B_2). Then $U \leqslant D_J$.*

Proof. The hypothesis guarantees that G_J is perfect (see [St, Lemma 32′]). Thus $U_{\alpha_j} \leqslant G_J \leqslant D_J$ for all $j \in J$. Furthermore $U_\alpha \leqslant D_J$ for all $\alpha \in \Sigma^+ \backslash \Delta$ by Lemma 2. Thus it suffices to show $U_{\alpha_i} \leqslant D_J$. Choose $j \in J$ with $(\alpha_j, \alpha_i) \neq 0$, and choose $n \in G_J$ representing $r := r_{\alpha_j}$. Then $r(\alpha_i) \in \Sigma^+ \backslash \Delta$, hence $U_{\alpha_i} = n^{-1} U_{r(\alpha_i)} n \leqslant n^{-1} D_J n = D_J$. ∎

Now let K be an algebraic closure of k. (In fact, our results remain true with K being any overfield of k, since k is a splitting field for G). By [St, Th. 43] the equivalence classes of irreducible KG-modules V are classified by dominant weights $\lambda = \sum_{i=1}^{l} m_i \lambda_i$ with $0 \leqslant m_i \leqslant q-1$. The λ corresponding to V will be called the highest weight of V. This correspondence is "inductive" in the following sense: For any $J \subset \Pi$ let V_J denote the fixed space in V of U_J. Then V_J is an irreducible module for G_J (by [S]), and if J is non-empty then the highest weight of V_J (as KG_J-module) is $\sum_{j \in J} m_j \lambda'_j$ (where the λ'_j $(j \in J)$ are the fundamental weights of Σ_J corresponding to Δ_J). If $J = \phi$, i.e., $P_J = B$, then $V_B := V_\phi = C_V(U)$ is 1-dimensional and T acts on it via λ.

From now on V denotes an irreducible KG-module with highest weight λ. We consider a non-splitting extension

$$0 \longrightarrow K \longrightarrow E \xrightarrow{\pi} V \longrightarrow 0$$

as in the Introduction. For each $J \subset \Pi$ let E_J denote the fixed space in E of U_J, and set $E_B := E_\phi = C_E(U)$. As explained in the Introduction, we want to study the "highest weight module condition" $(*)$ for E. Note that $(*)$ is equivalent to

$$\dim E_B > 1. \qquad (*)'$$

We are now ready to give our first criterion for $(*)$. Recall that $l = \operatorname{rank} G > 1$.

PROPOSITION 1. *Suppose λ is a multiple of a fundamental weight λ_i, and set $J := \Pi \backslash \{i\}$. If $q \leqslant 3$ (resp. $q = 2$) assume that the diagram of Δ_J has no isolated node (resp. no connected component of type B_2). Then $(*)$ holds for each non-splitting extension E of V over K.*

Proof. The parabolic P_J fixes the 1-space V_B $(= V_J)$ in V. Thus P_J also fixes the 2-space $Y := \pi^{-1}(V_B)$ in E, and acts triangularly in Y. Since the subgroup $G_J U_J$ of P_J is generated by p-elements, it acts in Y as a subgroup of

$$\left\{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in k\right\}.$$

So $D_J = (G_J U_J)'$ acts trivially in Y. But $U \leqslant D_J$ by Lemma 3, hence $Y \subset C_E(U) = E_B$. This proves$(*)'$, hence $(*)$. ∎

Remark. The additional hypothesis for small q cannot be dropped completely (see Remark 2 below). For fundamental λ, we will settle the cases excluded in Proposition 1. (Note that in these exceptions we have $q \leqslant 3$, hence $\lambda = \lambda_i$ or $\lambda = 2\lambda_i$, the latter only occurring for $q = 3$).

LEMMA 4. *If* $\dim E_J > 1$ *for some* $J \subset \Pi$, *we have the exact sequence*

$$0 \to K \to E_J \to V_J \to 0$$

of L_J*-modules.*

Proof. The hypothesis implies that $\pi(E_J)$ is a non-zero L_J-submodule of V_J, hence equals V_J (since V_J is irreducible under L_J). ∎

PROPOSITION 2. *Suppose* λ *is a fundamental weight* λ_i, *and exclude the cases*

(a) $q = 2$, $\Sigma \cong A_2$.

(b) $q = 2$, $\Sigma \cong A_3$, *and* $i = 2$.

(c) $q = 3$, $\Sigma \cong B_2$, *and* $i = 1$.

Then $(*)$ *holds for each non-splitting extension* E *of* V *over* K.

Proof. We assume the hypothesis of the Proposition. Additionally, we may assume that we are in one of the cases excluded in Proposition 1:

(1) Either $q \leqslant 3$ and the diagram of $\Delta_{\Pi \setminus \{i\}}$ has an isolated node, or $q = 2$ and the diagram has a connected component of type B_2.

The strategy of proof is to reduce to small rank cases, where we can apply the results of [JP]:

(2) Condition $(*)$ holds if λ is "minimal" in the sense of [JP].

Namely, in this case $H^1(G, V)$ is known: First, exclude the case $q = 2$, $\Sigma \cong C_l$ $(l \geqslant 3)$. Since we also excluded the cases (a), (b), (c), it follows from [JP, Sect. 3, 5, and 6C] that if λ is minimal then E is a quotient of the

Weyl module W_λ corresponding to V. But W_λ is generated by a B-semi-invariant, hence so is E.

Now assume $q=2$, $\Sigma \cong C_l$ $(l \geqslant 3)$. Then only λ_1 and λ_2 are minimal. If $\lambda=\lambda_2$ then again E is a quotient of W_λ (which follows by comparing [JP, 6B] and [CPS, table on p. 173]). If $\lambda=\lambda_1$ then we reduce to the case $\Sigma \cong B_l$, $\lambda=\lambda_1$ by the exceptional isomorphism $B_l(2) \to C_l(2)$ (cf. [CPS, p. 182]) and use that λ_1 is also minimal for type B_l (but not λ_2). This completes the proof of (2).

In the rest of the proof, we check all irreducible root systems individually. For the list of minimal weights, see [JP, 6B].

Case 1. $\Sigma \cong A_l$ $(l<5)$.

In this case all fundamental weights are minimal, hence (2) applies. (In fact, the restriction $l<5$ is unnecessary, but I want to stress that we need the results of [JP] only in small rank cases.)

Case 2. $\Sigma \cong G_2$.

In this case λ_2 is minimal. The case $\lambda=\lambda_1$ reduces to the case $\lambda=\lambda_2$ if $q=3$ (by the exceptional graph automorphism of $G_2(3)$). Finally, the case $\lambda=\lambda_1$, $q=2$ cannot occur here, since in this case $H^1(G, V^*)=H^1(G, V)=0$ (see the Appendix), contradicting the existence of non-splitting E.

Case 3. $\Sigma \cong B_2$.

Both fundamental weights are minimal, so again (2) applies.

Case 4. $\Sigma \cong C_3$.

By (1) we have $\lambda=\lambda_1$ or λ_2. But both λ_1 and λ_2 are minimal.

Case 5. $\Sigma \cong B_3$.

If $q=3$ then $\lambda=\lambda_2$ by (1); in this case again $H^1(G, V^*)=H^1(G, V)=0$ (see the Appendix). The case $q=2$ reduces to Case 4, since $B_3(2) \cong C_3(2)$.

Case 6. $\Sigma \cong F_4$.

By (1) we have $\lambda=\lambda_2$ or λ_3. First assume $\lambda=\lambda_2$. Since the group $G_{\{3,4\}} \cong A_2(q)$ is perfect, it acts trivially in $Y=\pi^{-1}(V_B)$ (by the same argument as in the proof of Proposition 1). Thus $U_{\alpha_4} \leqslant G_{\{3,4\}}$ acts trivially in Y. By Lemma 2, also U_α acts trivially in Y for each $\alpha \in \Sigma^+ \setminus \Delta$. Setting $\Omega := \{1, 2, 3\}$, it follows that U_Ω acts trivially in Y, hence $\dim E_\Omega \geqslant \dim Y > 1$. By Lemma 4, we get the extension

$$0 \to K \to E_\Omega \to V_\Omega \to 0 \tag{3}$$

of G_Ω- modules. Since $G_\Omega \cong B_3(q)$, we can invoke Case 5 to conclude that E_Ω satisfies the analogue of $(*)'$:

$$\dim C_{E_\Omega}(U \cap G_\Omega) > 1.$$

(Note that this trivially holds if E_Ω splits over K.)

Then clearly this centralizer equals Y, so now Y is centralized by $U \cap G_\Omega$. Above we saw that Y is centralized by U_Ω. It follows that Y is centralized by $U = (U \cap G_\Omega) U_\Omega$. Thus $Y \leqslant E_B$. This proves $(*)'$, hence $(*)$, if $\lambda = \lambda_2$.

The case $\lambda = \lambda_3$ is analogous (invoking Case 4 instead of Case 5).

Case 7. $\Sigma \cong A_l$ $(l \geqslant 5)$, D_l $(l \geqslant 4)$ or E_l $(l = 6, 7, 8)$.

In the case D_4 we have $\lambda = \lambda_2$ by (1), so λ is minimal. Now assume $\Sigma \neq D_4$. Then by (1), the diagram of $\Delta_{\Pi \setminus \{i\}}$ has exactly one or two isolated nodes; let $I' \subset \Pi$ be the set of j such that α_j corresponds to such an isolated node. Set $I := I' \cup \{i\}$. Then $G_{\Pi \setminus I}$ is perfect, and it follows as in Case 6 that U_I acts trivially in Y.

Now choose Ω with $I \subset \Omega \subset \Pi$ such that $\Sigma_\Omega \cong A_4$ or D_4. Then $\dim E_\Omega \geqslant \dim E_I > 1$, so again we get the extension (3) of G_Ω-modules. Since $G_\Omega \cong A_4(q)$ or $D_4(q)$, we can invoke Case 1 (resp. the above case of D_4) to conclude that E_Ω satisfies the analogue of $(*)'$. From this it follows as in Case 6 that E satisfies $(*)'$.

Case 8. $\Sigma \cong B_l$ or C_l $(l \geqslant 4)$.

By (1) we have $\lambda = \lambda_2$, λ_{l-2} or λ_{l-1}. The weight λ_2 is minimal for type C_4, and if $q = 2$ then the case $\Sigma \cong B_4$, $\lambda = \lambda_2$ reduces to that as above. So we can exclude the case $l = 4$, $\lambda = \lambda_2$, $q = 2$.

Now set $I := \{1, 2\}$ if $\lambda = \lambda_2$, $I := \{l-1, l\}$ if $\lambda = \lambda_{l-1}$, and $I := \{l-2, l-1, l\}$ if $\lambda = \lambda_{l-2}$ and $l > 4$. Then $G_{\Pi \setminus I}$ is perfect (by the exclusion in the previous paragraph). Thus $G_{\Pi \setminus I}$ acts trivially in Y, and so the same is true for each U_α with $\alpha \in \Sigma_{\Pi \setminus I}$. If $I = \{1, 2\}$ it follows by Lemma 1 that also for each $\alpha \in \Sigma^+ \setminus \Delta$, the group U_α acts trivially in Y; then also U_I acts trivially in Y. In the other two cases, the exceptional roots from Lemma 1(i), (i)$'$ lie in Σ_I, hence Lemma 1 implies that for all $\alpha \in \Sigma^+ \setminus (\Delta \cup \Sigma_I)$, the group U_α acts trivially in Y. Thus in all cases, U_I acts trivially in Y.

Now we choose Ω with $I \subset \Omega \subset \Pi$ such that $\Sigma_\Omega \cong A_2$ if $l = 4$, $\lambda = \lambda_2$, $q = 3$, $\Sigma_\Omega \cong A_4$ if $l > 4$, $\lambda = \lambda_2$, and $\Sigma_\Omega \cong B_3$ or C_3 otherwise. By the previous paragraph, $\dim E_\Omega \geqslant \dim E_I > 1$. Thus we get again the extension (3), which satisfies the analogue of $(*)'$ by Case 1, 4 and 5. Now it follows as in Case 6 (and 7) that E satisfies $(*)'$, hence $(*)$. ∎

Remark 1. In the cases (a), (b), (c), we have $\dim H^1(G, V^*) = 1$ by [JP], and *none* of the non-splitting extensions of V over K satisfies (*) (see Remark 2 below).

Condition (*) implies that E is a quotient of the induced module $I(V) := \mathrm{Ind}_B^G(V_B)$. This is not yet very helpful for the actual computation of $H^1(G, V^*)$, since the structure of $I(V)$ is not transparent. But in many cases, we can replace $I(V)$ by the much smaller module $\tilde{V}$, defined as follows: Let $f: I(V) \to V$ be the canonical KG-module map (extending the inclusion map $V_B \to V$). For each $j \in \Pi$, let M_j be the $G_{\{j\}}$-submodule of $I(V)$ generated by V_B. Now let M be the G-submodule of $I(V)$ generated by all the $M_j \cap \ker(f) (j \in \Pi)$, and define $\tilde{V}$ to be $I(V)/M$. (Then $\tilde{V} = H_0(\mathcal{U}_\lambda)$ in the set-up of [RS-2, 3B]; for the definition adopted here, see [R], where it is also proved that $\tilde{V}$ has a unique maximal submodule). More geometrically, $\tilde{V}$ can be viewed as the K-vector space abstractly generated by the G-conjugates of the 1-space V_B, subject to those relations that hold inside the G-conjugates of the spaces $V_{\{j\}}$. In fact, $\tilde{V}$ has explicitly been computed in several cases by geometric methods (see below).

LEMMA 5. *Assume (*) holds, and the extension*

$$0 \to K \to E_{\{j\}} \to V_{\{j\}} \to 0$$

of $G_{\{j\}}$-modules splits for every $j \in \Pi$ with $\dim V_{\{j\}} > 1$. *Then E is a quotient of $\tilde{V}$.*

Proof. First note that the above extension exists by Lemma 4, and splits automatically if $\dim V_{(j)} = 1$. (Namely in this case, $E_{\{j\}} = E_B$, and so $U \cap G_{\{j\}}$, hence $G_{\{j\}}$, acts trivially in $E_{\{j\}}$). Clearly, the extension splits also under $L_{\{j\}}$.

By (*) there is a KB-module map $\iota: V_B \to E$ such that $\pi\iota = \mathrm{id}_{V_B}$. (Recall that π is the map $E \to V$). We can choose ι such that

(1) For each $j \in \Pi$, the $G_{\{j\}}$-submodule $\bar{V}_{\{j\}}$ of $E_{\{j\}}$ generated by $\iota(V_B)$ splits $E_{\{j\}}$ over $\ker(\pi)$:

Namely, if $\lambda(T) \neq 1$ then $\iota(V_B)$ is the unique 1-dimensional KB-submodule of $E_{\{j\}}$ different from $\ker(\pi)$, hence (1) follows from the fact that $E_{\{j\}}$ splits under $L_{\{j\}}$. Now assume $\lambda(T) = 1$. Then $\lambda = \sum_{j=1}^{l} m_j \lambda_j$ with $m_j = 0$ or $q-1$ (since $1 = \lambda(h_{\alpha_j}(s)) = s^{\langle \lambda, \alpha_j \rangle} = s^{m_j}$ for each $s \in k^*$). Let J be the set of those j with $m_j = q-1$. Then J is non-empty (since otherwise $\lambda = 0$ and (*) forces B, hence G, to act trivially in E, which contradicts our standing hypothesis that E is non-splitting). Now V_J is the Steinberg module of G_J, hence is projective. So E_J splits over $\ker(\pi)$ as G_J-module. Thus we can choose ι so that $\iota(V_B)$ lies in the unique G_J-submodule of E_J isomorphic to V_J. Then (1) is clear for $j \in J$. But if $j \notin J$ then $G_{\{j\}}$ acts trivially in $E_{\{j\}}$, and so (1) holds again.

Let $h: I(V) \to E$ be the KG-module map extending ι. Then $\pi h = f$. Thus h maps $\ker(f)$ into $\ker(\pi)$. On the other hand, h maps M_j into $\bar{V}_{\{j\}}$ (by definition of these modules). By (1), it follows that h vanishes on $M_j \cap \ker(f)$ for each $j \in \Pi$. Thus the map $h: I(V) \to E$ vanishes on M, hence factors through $\tilde{V}$. ▮

Once (*) is known to hold, the hypothesis of Lemma 5 can easily be verified in many cases, since the extension theory for irreducible modules of $G_{\{j\}} \cong SL_2(q)$ is known [AJL]. For the case of fundamental modules, we need only the elementary fact that the 1-cohomology of $SL_2(q)$ in its natural module vanishes unless $q = 2^n > 2$:

COROLLARY 1. *Suppose $q = p$ is a prime, or λ is a fundamental weight. Then each non-splitting extension E of V over K satisfying $(*)$ is a quotient of $\tilde{V}$.*

Proof. If (*) holds for E, then the analogue of $(*)'$ holds for each $E_{\{j\}}$; hence if $E_{\{j\}}$ is non-splitting then it is a quotient of $\mathrm{Ind}_{G_{\{j\}} \cap B}^{G_{\{j\}}}(V_B)$. But it is a well known fact that a $K[SL_2(p)]$-module induced from a 1-dimensional module Z of a Borel subgroup has exactly two composition factors, neither of which is trivial unless Z is trivial; and if Z is trivial then the induced module is semisimple. Thus if $q = p$ is a prime, the extension in Lemma 5 splits for each $j \in \Pi$, proving the claim in this case.

Now suppose $\lambda = \lambda_i$ is fundamental. Then $V_{\{j\}} = V_B$ for each $j \neq i$, and $V_{\{i\}}$ is 2-dimensional, affording the natural module of $G_{\{i\}} \cong SL_2(q)$. Thus $H^1(G_{\{i\}}, V^*_{\{i\}}) \cong H^1(G_{\{i\}}, V_{\{i\}}) = 0$ unless $q = 2^n > 2$. By Lemma 5, it only remains to show that the extension

$$(1) \qquad 0 \to K \to E_{\{i\}} \to V_{\{i\}} \to 0$$

of $G_{\{i\}}$-modules splits for $q = 2^n > 2$.

Exclude for a moment the case that $q = 4$, $\Sigma \cong G_2$ and α_i is long. Then we have:

(2) There is $t \in T$ with $\lambda_i(t) \neq 1$, $\alpha_i(t) = 1$.

This implies the claim as follows: Since T acts in $V_{\{i\}}$ with weights $\lambda_i, \lambda_i - \alpha_i$ (restricted to T), it follows that t acts in $V_{\{i\}}$ as scalar multiplication by $\lambda_i(t) \neq 1$. Furthermore t centralizes $G_{\{i\}}$ (since $\alpha_i(t) = 1$). Hence the $\lambda_i(t)$-eigenspace of t in $V_{\{i\}}$ splits (1).

Now we prove (2) (under the hypothesis that $q = 2^n > 2$, and $q > 4$ if

$\Sigma \cong G_2$ and α_i is long). Choose $j \in \Pi \backslash \{i\}$ with $(\alpha_i, \alpha_j) \neq 0$, and take t of the form $t = h_{\alpha_i}(s_i) h_{\alpha_j}(s_j)$ with $s_i, s_j \in k^*$. Then

$$\lambda_i(t) = s_i^{\langle \lambda_i, \alpha_i \rangle} s_j^{\langle \lambda_i, \alpha_j \rangle} = s_i$$
$$\alpha_i(t) = s_i^{\langle \alpha_i, \alpha_i \rangle} s_j^{\langle \alpha_i, \alpha_j \rangle} = s_i^2 s_j^{-\varepsilon},$$

where $\varepsilon = -\langle \alpha_i, \alpha_j \rangle = 1, 2$, or 3. If $\varepsilon = 1$ or 2, then there is $s_j \in k^*$ with $s_j^\varepsilon \neq 1$ (since $q = 2^n > 2$). The case $\varepsilon = 3$ occurs only if $\Sigma \cong G_2$ and α_i is long; in this case we have $q > 4$, and so there is $s_j \in k^*$ with $s_j^\varepsilon = s_j^3 \neq 1$. Then in any case, we have $s_i \in k^*$ with $s_i^2 = s_j^\varepsilon \neq 1$, hence also $s_i \neq 1$ (since $p = 2$). Now $t = h_{\alpha_i}(s_i) h_{\alpha_j}(s_j)$ satisfies (2).

It remains to settle the case that $q = 4$, $\Sigma \cong G_2$ and α_i is long. Then V is the 14-dimensional adjoint module of $G_2(4)$; under $G_0 := \langle U_\alpha : \alpha \in \Sigma \text{ long} \rangle \cong A_2(4)$ it decomposes as $V = A \oplus V_3 \oplus \bar{V}_3$, where A is the (irreducible) 8-dimensional adjoint module of $A_2(4)$, and V_3, $\bar{V}_3$ are the natural 3-dimensional module and its dual (see the Appendix). Thus $H^1(G_0, V^*) \cong H^1(G_0, V) \cong H^1(G_0, A) \oplus H^1(G_0, V_3) \oplus H^1(G_0, \bar{V}_3) = 0$ (by [CPS]), hence E splits as KG_0-module. But $G_{\{i\}} \leqslant G_0$; so E splits under $G_{\{i\}}$, hence the same is true for $E_{\{i\}}$. ▮

THEOREM 1. *Suppose that rank $G > 1$, and the irreducible KG-module V is fundamental with highest weight λ_i, excluding cases* (a)–(c) *of Proposition 2. Then each non-splitting extension E of V over K is a quotient of $\tilde{V}$.*

Proof. Proposition 2 and Corollary 1.

Remark 2. In cases (a)–(c) we have $\tilde{V} \cong V$ by [RS-1], and $\dim H^1(G, V^*) = 1$ by [JP, 6C]. So these cases actually give rise to counterexamples.

Remark 3. $\tilde{V}$ has been calculated essentially for the modules V minimal in the sense of [RS-1] (in that paper) and for the adjoint modules not of type C_l (in [SV], [V-1]). In all these cases except for the adjoint module of type A_l, the module V is fundamental and so Theorem 1 yields $H^1(G, V^*)$ (with the exceptions from (a)–(c)). (We could easily adapt our method to include the adjoint module of A_l, but this does not seem worthwhile, since the 1-cohomology of this module is known by [JP]).

Most of these cohomology groups have already been calculated in [CPS] and [JP], except for the adjoint modules of tpe B_l, C_l, G_2, and F_4. An announced result of Hertzig [He] (proofs have never appeared) covers also the case of F_4.

In [V-2] the present author develops another approach to the 1-cohomology of the adjoint modules, which is more general in that it applies also to the case of infinite ground fields (as in [He]) and covers the case C_l as well. However, that approach breaks down for small fields ($q \leqslant 9$). By contrast, the methods of the present paper are primarily aimed at small fields.

Summarizing, we now know the 1-cohomology of all irreducible modules of adjoint type, except type C_l ($l>2$) for $q=3, 5, 9$. Thereby, by "irreducible module of adjoint type" we mean the irreducible KG-module V whose highest weight is the highest root of Σ (i.e., V is the unique irreducible quotient of the usual adjoint module).

COROLLARY 2. *Assume Σ is not of type A_l, C_l, or B_2. Then for the irreducible KG-module V of adjoint type, we have $H^1(G, V)=0$ unless $p=2$, $\Sigma \cong B_l$ ($l>2$ even) or $p=3$, $\Sigma \cong E_6$ or $p=2$, $\Sigma \cong E_7$ or $p=2$, $\Sigma \cong D_l$ ($l>3$). In these exceptional cases we have* $\dim H^1(G, V)=1$ *except in the last one for l even, where* $\dim H^1(G, V)=2$.

Proof. Let L denote the usual adjoint module of G, i.e., the Lie algebra of the (simply-connected) algebraic group G_K enclosing G (whereby G acts on L in the restriction of the adjoint representation of G_K). Then V is the unique irreducible quotient of L (and L is the Weyl-module corresponding to V); by [V-1, Cor. 2] the module $\tilde{V}$ is isomorphic to L (since Σ is not of type C_l or B_2). (Strictly speaking, in [V-1] the analogue of $\tilde{V}$ over k (instead of K) is studied, but the transition is immediate).

The highest weight of V is the highest root of Σ, hence is fundamental in the cases under consideration (see [Bou, appendix]). Also $V \cong V^*$, hence by Theorem 1 the dimension of $H^1(G, V)$ can be read off from $\tilde{V} \cong L$: It equals the multiplicity of the trivial representation in $M/\mathrm{rad}(M)$, where M is the unique maximal submodule of L. This multiplicity is given by [H] (and [Hur]), where the full submodule structure of L is determined. ∎

3. MORE ON THE CONDITION (*)

In this section, $l = \mathrm{rank}\, G \geqslant 1$ (so we now include the case $l=1$). Otherwise, we keep all notations and conventions from Sections 1 and 2. In particular, V is an irreducible KG-module with highest weight λ, and

$$0 \longrightarrow K \longrightarrow E \xrightarrow{\pi} V \longrightarrow 0$$

is a non-splitting extension of KG-modules. We set $Y := \pi^{-1}(V_B)$ (a 2-dimensional subspace of E).

LEMMA 6. *If U_α acts non-trivially in Y for some $\alpha \in \Sigma$, then*

$$\lambda + p^r\alpha \in (q-1)X$$

for some integer r with $0 \leqslant r \leqslant n-1$.

Proof. Fix $\alpha \in \Sigma$, and pick a basis y, z of Y consisting of weight vectors for T, with $z \in \ker(\pi)$. Then y is a vector of weight λ. Furthermore, there is an endomorphism f of the additive group of k with $x_\alpha(a) \cdot y = y + f(a)z$ for each $a \in k$. Since we assume that U_α acts non-trivially in Y, we have $f \neq 0$.

For all $t \in T$, $a \in k$ we get $y + f(\alpha(t)a)z = x_\alpha(\alpha(t)a) \cdot y = [tx_\alpha(a)t^{-1}] \cdot y = [tx_\alpha(a)] \cdot \lambda(t^{-1})\, y = y + \lambda(t)^{-1} f(a)z$, thus

$$f(\alpha(t)a) = \lambda(t)^{-1} f(a) \qquad \text{for all } t \in T, \quad a \in k. \tag{1}$$

If $f(1) = 0$ then $f(s^2) = f(\alpha(h_\alpha(s)) = \lambda(h_\alpha(s))^{-1} f(1) = 0$ for all $s \in k^*$; thus $f = 0$, since $f(a) = f((a/2+1)^2) - f(a^2/4) - f(1^2) = 0$ if $p \neq 2$, and $k = k^2$ if $p = 2$. But we have $f \neq 0$, hence $f(1) \neq 0$. This allows us to assume $f(1) = 1$ (replacing z by $f(1)z$). Then $f(\alpha(t)) = \lambda(t)^{-1} f(1) = \lambda(t)^{-1}$, hence $f(\alpha(t)a) = \lambda(t)^{-1} f(a) = f(\alpha(t)) f(a)$ for all $t \in T$, $a \in k$. Thus

$$f(ab) = f(a) f(b) \tag{2}$$

if either a or b is a square. In fact, (2) holds for all $a, b \in k$: This is clear if $p = 2$; if $p \neq 2$, then $f(ab) = f([(a/2+1)^2 - a^2/4 - 1]b) = f((a/2+1)^2) f(b) - f(a^2/4) f(b) - f(1) f(b) = f(a) f(b)$.

Thus f is a ring homomorphism, and since $f \neq 0$ it follows that f is an automorphism of k. Since k is the field with $q = p^n$ elements, there is r with $0 \leqslant r \leqslant n-1$ such that $f(a) = a^{p^r}$ for all $a \in k$. Now (1) gives $\lambda(t)^{-1} = f(\alpha(t) = \alpha(t)^{p^r}$. Thus $\lambda + p^r\alpha$ vanishes on T, hence lies in $(q-1)X$. ∎

The extension theory for irreducible modules of $SL_2(q)$ is known by [AJL]. In the case $G = SL_2(q)$, this allows us to determine all extensions E satisfying $(*)$. This will be the base for the higher rank cases.

LEMMA 7. *Suppose $G = SL_2(q)$ (i.e., $l = 1$).*

(1) *If $q = p$ then $(*)$ never holds.*

(2) *If $q \neq p$ then $(*)$ always holds.*

Furthermore, in the case $q = p > 2$ the non-splitting extension E can only exist if $\lambda = (p-3)\lambda_1$.

Proof. (1) See the case $q = p$ of the proof of Corollary 1.

(2) We have $\lambda = m\lambda_1$ with $0 \leqslant m \leqslant q-1$. Since we are now in the case $q = p^n \neq p$, it follows from [AJL, Cor. 4.5] that the non-splitting extension E can only exist for the following values of m:

$$1+(p-2)p^{n-1}, \quad p^s((p-2)+p) \qquad \text{for} \quad 0 \leqslant s \leqslant n-2.$$

We first consider the latter case. In this case V is the sth-Frobenius twist of the irreducible module with highest weight $((p-2)+p)\lambda_1$, hence we may assume $s=0$. Then by Steinberg's tensor product theorem, $V \cong V_{p-2} \otimes V_1^{(p)}$, where V_{p-2} is the irreducible module with highest weight $(p-2)\lambda_1$ and $V_1^{(p)}$ is the Frobenius twist of the 2-dimensional natural module. Thus $\dim V = 2(p-1)$. On the other hand, the Weyl module W_λ corresponding to V has dimension $m+1 = 2p-1$. Thus W_λ is a non-splitting extension of V over K (cf. [W, Th. 2D]). Since $\dim \mathrm{Ext}^1_{KG}(V, K) = 1$ by [AJL, Cor. 4.5], each such non-splitting extension E is isomorphic to W_λ. But W_λ satisfies $(*)$, so we are done.

To deal with the remaining case $m = 1+(p-2)p^{n-1}$, we assume by way of contradiction that $(*)$ does not hold. Then $U = U_{\alpha_1}$ acts non-trivially in Y, hence by Lemma 6 we have $\lambda + p^r\alpha_1 \in (q-1)X$ for some r with $0 \leqslant r \leqslant n-1$. Since $\alpha_1 = 2\lambda_1$, we get $m + 2p^r \mid (q-1)$. Then from $m + 2p^r = p^n - 2p^{n-1} + 2p^r + 1 \leqslant p^n + 1$ it follows that $m + 2p^r = q-1$, hence $2p^{n-1} - 2p^r = 2$. So $p^{n-1} - p^r = 1$, which is only possible if $p=2$, $n=2$, $r=0$. In this case, $\dim H^1(G, V) = \dim H^1(SL_2(4), K^2) = 1$, hence the Frobenius twist of E isomorphic to the module of traceless 2×2-matrices, which is a non-splitting extension of $V_1^{(2)}$ over K satisfying the analogue of $(*)$ (cf. [CPSK, 7.6(a)]). This settles the case $p=2$, $n=2$, $r=0$, and thereby case (2).

The last assertion in the Lemma follows from the known (elementary) extension theory for irreducible modules of $SL_2(p)$, see e.g., [A1, p. 48–49]. (It also follows from Lemma 6). ∎

THEOREM 2. *Suppose that either q is not a prime, or $q > 2$ and $\langle \lambda, \alpha_j \rangle \neq q-3$ for all $j \in \Pi$. Assume $q > 3$ if $\Sigma \cong G_2$. Then $(*)$ holds for each non-splitting extension E of V over K.*

Proof. In view of the extra hypothesis in the case G_2, Lemma 1 yields $U_\alpha \leqslant U'$ for each $\alpha \in \Sigma^+ \setminus \Delta$; hence U_α acts trivially in Y. It remains to show that also the U_{α_j} act trivially in Y.

So fix $j \in \Pi$. We consider the extension

$$0 \to K \to \pi^{-1}(V_{\{j\}}) \to V_{\{j\}} \to 0 \tag{A}$$

of $G_{\{j\}}$- modules. By Lemma 7, this extension always splits in the case $q = p > 2$, since then $V_{\{j\}}$ is the irreducible module for $G_{\{j\}} \cong SL_2(p)$ of highest weight $m_j\lambda_j'$ and $m_j := \langle \lambda, \alpha_j \rangle \neq p-3$. But if (A) splits, then clearly U_{α_j} acts trivially in Y.

If $q \neq p$ and the extension (A) does not split, then it satisfies the analogue of $(*)$ by Lemma 7, which again means that U_{α_j} acts trivially in Y. Thus we have shown that U_{α_j} acts trivially in Y in any case, which completes the proof of the theorem. ▮

Remark. The following counterexample shows that the additional hypothesis in the case $q=p$ is in fact necessary for each prime $p>3$ (for $p \leqslant 3$, see Remark 2): If $q=p>3$, $\Sigma \cong A_2$ and $\lambda=(p-3)\lambda_1+\lambda_2$, then there exists a non-splitting extension of V over K not occurring in the induced module λ_B^G (hence not satisfying $(*)$), as remarked in [An, 4.2].

Using Lemma 6 instead of Lemma 7 in the proof of Theorem 2, we get the following

COROLLARY (OF PROOF). *Suppose $q=p>2$, and $q>3$ for type G_2. If*

$$\lambda+\alpha_j \notin (q-1)X$$

for all $j \in \Pi$, then $(*)$ holds for all non-splitting extensions E of V over K.

As in Section 1, our criteria for the condition $(*)$ can be combined with Lemma 5 (and Corollary 1) to yield conditions ensuring that E is a quotient of $\tilde{V}$.

APPENDIX

Here we provide some computations needed in the proof of Proposition 2. We show $H^1(G, V)=0$ if $G=G_2(2)$ (resp. $G=B_3(3)$) and V is the irreducible KG-module of highest weight λ_1 (resp. λ_2). In these cases, V is the unique irreducible quotient of the adjoint module L (as in the proof of Corollary 2), and then even $V=L$ (by [H]).

L comes with a Chevalley basis $\{E_\alpha, H_\beta : \alpha \in \Sigma,\ \beta \in \Delta\}$, such that the action of the generators $x_\alpha(a)$ of G on L is given in terms of this basis by the usual formulas (see, e.g. [H, Sect. 3]). Setting $L_\alpha := KE_\alpha$, $L_0 := \sum_{\beta \in \Delta} KH_\beta$, we have in particular:

$$(x_\alpha(a)-1)L_\beta \subset \sum_{i>0} L_{i\alpha+\beta} \qquad \text{for all } \alpha, \beta \in \Sigma, \quad a \in k. \tag{1}$$

(Thereby of course $L_{i\alpha+\beta}=0$ if $i\alpha+\beta \notin \sum \cup 0$).

For simplicity we write $x^\alpha := x_\alpha(1)$ for $\alpha \in \Sigma$.

LEMMA A. *Suppose $G=A_1(q)$ with $q=2$ or q odd, and $\Sigma=\{\alpha, -\alpha\}$. If* *c is a cocycle of G in its adjoint module L with* $c(U_\alpha)=0$ *and* $c(U_{-\alpha})\subset L_{-\alpha}$ *then* $c=0$.

Proof. First assume q is odd. Then $H^1(A_1(q), L)=0$, hence c is a coboundary, i.e., $c(g)=(g-1)v$ for some $v\in L$. Write $v=tE_\alpha+rE_{-\alpha}+sH_\alpha$ with $t, r, s\in k$. Then $0=c(x^\alpha)=(x^\alpha-1)v=rH_\alpha-(2s+r)E_\alpha$. Hence $r=s=0$. Similarly, the condition $c(x^{-\alpha})\subset L_{-\alpha}$ yields $t=0$. So $c=0$.

Now assume $q=2$. Then $w:=x^\alpha x^{-\alpha}x^\alpha$ is an involution, hence $w\cdot c(w)=c(w)$. We have $c(x^{-\alpha})=aE_{-\alpha}$ for some $a\in K$, and so we get $c(w)=aH_\alpha+aE_\alpha$. But $c(w)$ is fixed by w, which forces $a=0$, hence $c=0$. ∎

LEMMA B. *Suppose q is prime, and G fixes no vector* $\neq 0$ *in L. If c is a cocycle of G in L with* $c(U_\alpha)\subset L_\alpha$ *for all* $\alpha\in\Sigma$, *then c is a coboundary.*

Proof. First we consider the special case that additionally $c(U_\alpha)=0$ for all $\alpha\in\Delta$. Then also $c(U_{-\alpha})=0$ (by Lemma A, applied to the subgroup $\langle U_\alpha, U_{-\alpha}\rangle\cong SL_2(q)$), hence $c=0$ (since $G=\langle U_\alpha : \alpha\in\pm\Delta\rangle$).

For $H=\sum_{\beta\in\Delta}a_\beta H_\beta$ ($a_\beta\in K$), we have

$$(x^\alpha-1)H=\left(-\sum_{\beta\in\Delta}a_\beta c_{\beta\alpha}\right)E_\alpha \qquad \text{for all } \alpha\in\Sigma,$$

where the $c_{\beta\alpha}$ are the Cartan integers (see [H]). If the matrix $(c_{\beta\alpha})_{\alpha,\beta\in\Delta}$ were singular mod p, then there would be $H\neq 0$ as above with $(x^\alpha-1)H=0$ for all $\alpha\in\Delta\cup-\Delta$, hence H would be fixed by G, contradicting the hypothesis. So the matrix is non-singular mod p, which means we can choose H such that $(x^\alpha-1)H=c(x^\alpha)$ for all $\alpha\in\Delta$. Adjusting c by the coboundary $g\to(g-1)H$ brings us now back into the above special case. ∎

Set $G_0:=\langle U_\alpha : \alpha\in\Sigma \text{ long}\rangle$. Then $G_0\cong A_2(2)$ (resp. $G_0\cong A_3(3)$) if $G=G_2(2)$ (resp. $G=B_3(3)$). Let A denote the adjoint module of G_0, naturally embedded into L (i.e., $A=L_0+\sum_{\alpha\text{long}}L_\alpha$). Using (1) it is easy to see that L decomposes under G_0 as

$$L=A\oplus V_3\oplus\bar{V}_3 \qquad \text{resp.} \qquad L=A\oplus V_6, \tag{2}$$

where V_3, $\bar{V}_3$ are the natural 3-dimensional module of $A_2(2)$ and its dual; furthermore V_6 is the 6-dimensional orthogonal module of $A_3(3)\cong\Omega_6^+(3)$. (Clearly, $V_6=\sum_{\alpha\text{short}}L_\alpha$, and V_3, $\bar{V}_3$ are each spanned by the L_α where α runs through an orbit of short roots under the Weyl group of G_0).

Now we fix a cocycle c of G in L. If $G=B_3(3)$ then $H^1(G_0, L)\cong H^1(G_0, A)\oplus H^1(G_0, V_6)=0$ by [JP], hence we may assume $c|_{G_0}=0$. For

$G = G_2(2)$, this will require some more work later. For now, we just assume this condition

$$c\,|_{G_0} = 0. \tag{3}$$

Pick a short $\alpha \in \Sigma$, and let C_0 (resp. P_0) denote the centralizer (resp. normalizer) of L_α in G_0. Since L_α is a 1-space in V_3 or $\bar{V}_3$ (resp. an isotropic 1-space in V_6), P_0 is a maximal parabolic subgroup of G_0; it follows from (2) that P_0 fixes no other 1-space in L. Thus L_α is the full centralizer of C_0 in L (since this centralizer is invariant under P_0, and $P_0/C_0 \cong k^*$). Clearly, C_0 centralizes also U_α (e.g., use the uniqueness in [Hu, Th. 26.3(a)]); by (3) (and (4) below) this implies that C_0 centralizes $c(U_\alpha)$, hence $c(U_\alpha) \subset C_L(C_0) = L_\alpha$. Thus $c(U_\alpha) \subset L_\alpha$ for all short roots $\alpha \in \Sigma$; but this trivially holds also for the long roots (by (3)). Hence Lemma B applies, proving that c is a coboundary.

It remains to justify hypothesis (3) in the case $G = G_2(2)$. We proceed similarly as in [J, Ch. 9]. First we record a useful formula from [J, 3.3] (which follows directly from the cocycle condition):

For commuting elements $g_1, g_2 \in G$ we have $(g_1 - 1)c(g_2) = (g_2 - 1)c(g_1)$.
$$\tag{4}$$

From now on $G = G_2(2)$, and we fix a basis γ, δ of Σ with γ long. Since $H^1(G_0, A) = 0$ (by [JP]), we may assume by (2) that $c(G_0) \subset V_3 \oplus \bar{V}_3$. So $c(x^\gamma) = \sum_{\alpha\,\text{short}} a_\alpha E_\alpha$ for certain $a_\alpha \in K$. Since x^γ is an involution, we have $x^\gamma \cdot c(x^\gamma) = c(x^\gamma)$. This implies $a_\delta = a_{-\gamma-\delta} = 0$ (using (1)).

Let c' be the coboundary defined by $c'(g) = (g-1)\cdot(a_{\gamma+\delta}E_\delta + a_{-\delta}E_{-\gamma-\delta})$, and set $c'' := c + c'$. Then

$$\begin{aligned} c''(x^\gamma) &= c(x^\gamma) + (x^\gamma - 1)\cdot a_{\gamma+\delta}E_\delta + (x^\gamma - 1)\cdot a_{-\delta}E_{-\gamma-\delta} \\ &= \left(\sum_{\alpha\,\text{short}} a_\alpha E_\alpha\right) + a_{\gamma+\delta}E_{\gamma+\delta} + a_{-\delta}E_{-\delta} \\ &= a_{\gamma+2\delta}E_{\gamma+2\delta} + a_{-\gamma-2\delta}E_{-\gamma-2\delta}. \end{aligned}$$

Thus, replacing c by c'', and setting $\eta := \gamma + 2\delta$, we may assume

$$c(x^\gamma) = aE_\eta + bE_{-\eta} \qquad (a, b \in K). \tag{5}$$

For later use we note that if for some long $\gamma' \in \Sigma$ with $(\gamma, \gamma') < 0$ the original c satisfied $c(x^{\gamma'}) = 0$, then this condition still holds after adjusting with the coboundary c' (since $c'(x^{\gamma'}) \subset (x^{\gamma'} - 1)\cdot(L_\delta \oplus L_{-\gamma-\delta}) = 0$ by (1)). By (4) and (5) we get:

$$\begin{aligned} (x^\gamma - 1)\cdot c(x^\eta) = (x^\eta - 1)\cdot c(x^\gamma) &= a(x^\eta - 1)E_\eta \\ &\quad + b(x^\eta - 1)E_{-\eta} = bH_\eta + bE_\eta \end{aligned}$$

If $b \neq 0$, it follows by (1) that

$$H_\eta + E_\eta \in (x^\gamma - 1)L \subset L_0 \oplus L_\gamma \oplus L_{\gamma+\delta}$$
$$\oplus L_{-\gamma-\delta} \oplus L_{2\gamma+3\delta} \oplus L_{-2\gamma-3\delta}.$$

This contradiction proves $b = 0$. Similarly, we get $a = 0$ (replacing η by $-\eta$ in the above). Now (5) gives:

$$c(x^\gamma) = 0. \tag{6}$$

Since $\gamma' := \gamma + 3\delta$ is long, we can also adjust c such that:

$$c(x^{\gamma'}) = 0. \tag{7}$$

As noted above, we can even get (6) and (7) simultaneously. Repeating this argument once more for $\gamma'' := \gamma - 3\delta$, we see that we can finally assume c to vanish on x^γ, $x^{\gamma'}$ and $x^{\gamma''}$. But these elements generate $G_0 \cong SL_3(2)$, and so $c\,|_{G_0} = 0$. This justifies hypothesis (3), and so the proof is complete.

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